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A Continuous Algorithm for Max-bisection*

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Abstract: A continuous algorithm for max-bisection is proposed. We first convert the max-bisection problem to a nonlinear program, then the resulted problem is solved by using the augmented Lagrange penalty function method.

Keywords: combinatorial optimization; max-bisection; penalty function; NCP

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1 Introduction

Let $G(V;E)$ be an undirected weighted graph, where $V=\{1,\cdots,n\}$ is the set of nodes, E is the set of edges. Let $W=(w_{ij})$ be its weighted adjacency matrix with $w_{ij}>0$ for $(i,j)\in E$ and $w_{ij}=0$ for $(i,j)\notin E$. Max-bisection consists in finding a partition of the set V into two sets S, $T=V\setminus S$, such that $|S|=|T|=n/2$ and the total weight of edges with endpoints in different subsets is maximized, i.e.

$$w(S)=\max\sum_{i\in S,j\in T}w_{ij},$$

when $n=|V|$ is even. The problem can be formulated by assigning each node a binary variable $x_i\in\{-1,1\}$

$$\text{MB}:\quad \max\Big\{f(\mathbf{x})=\frac{1}{4}\sum_{i,j}w_{ij}(1-x_ix_j)=\mathbf{x}^TL\mathbf{x}:\ \mathbf{e}^T\mathbf{x}=0,\ \mathbf{x}\in\{-1,1\}^n\Big\},$$

where

$$\mathbf{x}=(x_1,x_2,\cdots,x_n)^T,\quad S=\{i|x_i=1\},\quad \mathbf{e}=(1,1,\cdots,1)\in\mathbf{R}^n,\quad L=\frac{1}{4}(\mathrm{Diag}(W\mathbf{e})-W),$$

$\mathrm{Diag}(\mathbf{a})$ means a diagonal matrix whose diagonal entries are vector $\mathbf{a}$. The constraint $\mathbf{e}^T\mathbf{x}=0$ ensures that

$$|S|=|T|=\frac{n}{2}.$$

Frieze & Jerrum[1] extended Goemans-Williamson's approach[2] to max-bisection and obtianed an 0.651-approximation algorithm. Ye in [4] improved this result to 0.699. Recently, Xu et al.[3] proposed a feasible direction algorithm without linear search in which max-bisection was relaxed into a quadratic programming problem with nonlinear constraints

$$XMB:\ \max\{\mathbf{x}^TL\mathbf{x}:\Phi_F(1-x_j,1+x_j)\le 0,\ j=1,\cdots,n;\ \mathbf{e}^T\mathbf{x}=0;\ \|\mathbf{x}\|_2=1\},$$

Received date: 2007-10-16. **Biography**: Li Yu (Born in 1963), Male, Ph.D. Research field: combinatorial optimization and its application in financial optimization.

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where $\Phi_F(a,b)=\sqrt{a^2+b^2}-(a+b)$, $a,\ b\in\mathbf{R}$ is the FB function.

2 Equivalent nonlinear program

A function $\Phi:\mathbf{R}^2\to\mathbf{R}$ is called an NCP-function if $\Phi(a,b)=0\Leftrightarrow ab=0,\ a\geq 0,\ b\geq 0$. By this definition, we define $\Phi(a,b)=\max(0,ab)^3+\max(0,-a)^3+\max(0,-b)^3$. Clearly, $\Phi(a,b)$ is a NCP function. Let $a=1-x_i,\ b=1+x_i,\ i=1,\cdots,n$, then $\Phi(1-x_i,1+x_i)=0\Leftrightarrow x_i\{-1,1\}$. Moreover, it is easy to show $\Phi(1-x_i,1+x_i)$ has up to the second order derivation with respective to $x_i,\ i=1,\cdots,n$. Hence, MB can be rewritten as

$$\text{MB1}:\quad \max\{f(\mathbf{x})=\mathbf{x}^TL\mathbf{x}:\ \mathbf{e}^T\mathbf{x}=0;\ \Phi(1-x_i,1+x_i)=0,\ i=1,\cdots,n\}.$$

Let $\lambda_{\max}(L)$ denote the maximum eigenvalue of matrix L, and $\rho=\frac{1}{4}n\lambda_{\max}(L)$. Consider the following problem

$$\text{MB2}:\quad \max\{F(\mathbf{x})=\mathbf{x}^TL\mathbf{x}-\rho\mathbf{x}^T\mathbf{e}\mathbf{e}^T\mathbf{x}=\mathbf{x}^T(L-\rho\mathbf{e}\mathbf{e}^T)\mathbf{x}:\ \Phi(1-x_i,1+x_i)=0,\ i=1,\cdots,n\}.$$

Then we have the following results.

Lemma 2.1 Let $\mathbf{x}^{**}$ be an optimization solution of problem MB2. Then $\mathbf{x}^{**}$ is also an optimization solution of problem MB1, also MB2 and MB1 have the same optimization value at $\mathbf{x}^{**}$.

Lemma 2.1 shows that, instead of solving problem MB1 directly, we can also obtain its optimization solution by solving problem MB2. Let $C=-(L-\rho\mathbf{e}\mathbf{e}^T)$. Then MB2 can be written as

$$\text{MB3}:\quad \min\{h(\mathbf{x})=\mathbf{x}^TC\mathbf{x}=\mathbf{x}^T(\rho\mathbf{e}\mathbf{e}^T-L)\mathbf{x}:\ \Phi(1-x_i,1+x_i)=0,\ i=1,\cdots,n\}.$$

The augmented Lagrange penalty function method (ALPFM) of problem MB3 has the form

$$\mathcal{P}(\mathbf{x};\lambda,\sigma)=h(\mathbf{x})-\lambda^T\Psi(\mathbf{x})+\frac{\sigma}{2}\|\Psi(\mathbf{x})\|^2,$$

where

$$\Psi(\mathbf{x})=(\psi(x_1),\cdots,\psi(x_n))^T=(\Phi(1-x_1,1+x_1),\cdots,\Phi(1-x_n,1+x_n))^T,$$

$\lambda=(\lambda_1,\cdots,\lambda_n)^T$ is the Lagrange multiplier vector, and σ is a penalty factor. The function $\mathcal{P}(\mathbf{x};\lambda,\sigma)$ has up to second order derivative and the gradient of the function $\mathcal{P}(\mathbf{x};\lambda,\sigma)$ with respective to $\mathbf{x}$ is easily evaluated by

$$\nabla_{\mathbf{x}}\mathcal{P}(\mathbf{x};\lambda,\sigma)=2C\mathbf{x}-\nabla\Psi(\mathbf{x})(\lambda-\sigma\Psi(\mathbf{x})),$$

where

$$\nabla\Psi(\mathbf{x})=\mathrm{Diag}\Big(\frac{d\psi(x_1)}{dx_1},\cdots,\frac{d\psi(x_n)}{dx_n}\Big).$$

3 Solving problem MB3

Let $\mathbf{x}^*$ and λ^* be an optimal solution of problem MB2 and the associated Lagrangian multiplier vector, respectively. The KKT condition at $(\mathbf{x}^*, \lambda^*)$ of problem MB2 can be presented as

$$\nabla F(\mathbf{x}^*) - \nabla\Psi(\mathbf{x}^*)\lambda^* = 0. \tag{1}$$

Let $\{\lambda^k\}_{k=0}^{\infty}$ be a given sequence. For each given λ^k, we can minimize $\mathcal{P}(\mathbf{x}; \lambda^k, \sigma)$ for large enough σ. Assume that $\mathbf{x}^{k+1}$ is a minimizer of $\mathcal{P}(\mathbf{x}; \lambda^k, \sigma)$, then the sequence $\{\lambda^k\}_{k=0}^{\infty}$, for large enough σ, is modified by

$$\lambda^{k+1} = \lambda^k - \sigma\Psi(\mathbf{x}^{k+1}), \quad k = 0, 1, \cdots.$$

Hence, the minimizer $\mathbf{x}^*$ of MB2 can be obtained by minimizing the function sequence $\mathcal{P}(\mathbf{x}; \lambda^k, \sigma)$. Summarize the statement above, we can described the algorithm for MB3 as follows.

Algorithm CMB:

1) Choose randomly an initial point $\mathbf{x}^0 \in (-a, a)^n$, $a \geq 1$, given σ_0, λ^0, $0 < \epsilon < 1$ and $0 < \rho < 1$, set $k = 0$;
2) Start from $\mathbf{x}^k$, minimize $\mathcal{P}(\mathbf{x}; \lambda^k, \sigma_k)$ to obtain a solution $\mathbf{x}^{k+1}$. If $\| \Psi(\mathbf{x}^{k+1}) \| \leq \epsilon$, set $\mathbf{x}^* = \mathbf{x}^{k+1}$, goto 5);
3) If $\| \Psi(\mathbf{x}^{k+1}) \| > \rho \| \Psi(\mathbf{x}^k) \|$, set $\sigma_k = 10\sigma_k$ and $\mathbf{x}_k = \mathbf{x}_{k+1}$, goto 2);
4) If $\| \Psi(\mathbf{x}^{k+1}) \| \leq \rho \| \Psi(\mathbf{x}^k) \|$, set $\lambda^{k+1} = \lambda^k - \sigma_k\Psi(\mathbf{x}^{k+1})$, $\sigma_{k+1} = \sigma_k$, and $k = k+1$, goto 2);
5) Construct a bisection.
   (a) If $|S| = \frac{n}{2}$, then output S is an optimization bisection partition and stop;
   (b) If $|S| > \frac{n}{2}$, then for each node $i \in S$, compute
   $$\delta(i) = \sum_{j\in S} w_{ij} - \sum_{j\in T} w_{ij},$$
   and rearrange the set $S = \{i_1, i_2, \cdots, i_{|S|}\}$, where $\delta(i_1) \leq \delta(i_2) \leq \cdots \leq \delta(i_{|S|})$. Let $S = \{i_1, i_2, \cdots, i_{\frac{n}{2}}\}$;
   (c) If $|S| < \frac{n}{2}$, then set $\mathbf{x}^* = -\mathbf{x}^*$, goto step 5)(b).

The solution sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ obtained by the algorithm CMB generates a sequence $\{\Psi(\mathbf{x}^k)\}_{k=0}^{\infty}$ that converges to zero at a rate less than ρ, the condition $\| \Psi(\mathbf{x}^{k+1}) \| \leq \epsilon$ can be satisfied in a finite number of iterations (see [3]).

Theorem 3.1 If $\{\lambda^k\}_{k=0}^{\infty}$ is a bounded sequence and $\mathbf{x}^k$ is a global minimizer of $\mathcal{P}(\mathbf{x}; \lambda^{k-1}, \sigma_k)$, then any accumulation point $\mathbf{x}^*$ of the infinite sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ generated by the algorithm CMB with termination criterion $\| \Psi(\mathbf{x}^k) \| = 0$ is an optimal solution of problem MB3.

Proof *By algortihm CMB, the infinite sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ generated by the algorithm CMB* satisfies

$$\|\Psi(\mathbf{x}^{k+1})\| \leq \rho\|\Psi(\mathbf{x}^k)\|, \quad 0 < \rho < 1.$$

That is

$$\lim_{k\to\infty} \|\Psi(\mathbf{x}^k)\| = 0.$$

Hence, any accumulation point of the infinite sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ must be a feasible solution of MB3.

Let $\{\mathbf{x}^{k_j}\}$ be a convergent subsequence of $\{\mathbf{x}^k\}_{k=0}^{\infty}$ and $\mathbf{x}^*$ its limitation point. Then $\mathbf{x}^*$ is a feasible solution of MB3. Let $\mathbf{x}^{**}$ be a global optimal solution of MB3. Since $\mathbf{x}^{k_j}$ is a global minimizer of function $\mathcal{P}(\mathbf{x};\lambda^{k_j-1},\sigma_{k_j})$ and $\mathbf{x}^{**}$ is only a feasible solution of $\mathcal{P}(\mathbf{x};\lambda^{k_j-1},\sigma_{k_j})$, we have, for sufficient large k_j,

$$\begin{aligned} \mathcal{P}(\mathbf{x}^{k_j};\lambda^{k_j-1},\sigma_{k_j}) &= h(\mathbf{x}^{k_j}) - (\lambda^{k_j-1})^T\Psi(\mathbf{x}^{k_j}) + \frac{\sigma_{k_j}}{2} \parallel \Psi(\mathbf{x}^{k_j} \parallel^2 \\ &\le \mathcal{P}(\mathbf{x}^{**};\lambda^{k_j-1},\sigma_{k_j}) = h(\mathbf{x}^{**}). \end{aligned} \tag{2}$$

Since $\{\lambda^k\}_{k=0}^{\infty}$ is a bounded sequence, for sufficient k, it follows from (2) that

$$h(\mathbf{x}^{k_j}) + o(1) \le h(\mathbf{x}^{**}). \tag{3}$$

Let $k_j \to \infty$ in the left hand side of (3), it follows that

$$h(\mathbf{x}^*) = \lim_{k\to\infty} h(\mathbf{x}^{k_j}) \le h(\mathbf{x}^{**}).$$

On the other hand, by the feasibility of $\mathbf{x}^*$, $h(\mathbf{x}^*) \ge h(\mathbf{x}^{**})$. Therefore $h(\mathbf{x}^*) = h(\mathbf{x}^{**})$ which means $\mathbf{x}^*$ is an optimal solution of MB3.

4 Numerical experiments

For minimizing $\mathcal{P}(\mathbf{x};\lambda^{k-1},\sigma_k)$ with fixed λ^{k-1}, σ_k, we use the function *fminunc* in Matlab.

The test problems are some randomly generated graphs. Let $p \in (0,1)$ be fixed. For each pair (i,j), $i \ne j$, $c \in (0,1)$ is random number. If $c \le p$, then there is an edge between nodes i and j, and set w_{ij} is a random integer number in [1,10]. Otherwise, set $w_{ij} = 0$. We compare our algorithm with the 0.699-approximation algorithm, see Table 1. From the numerical results, one is not hard to see that the proposed continuous algorithm can generate a better solution than the 0.699-approximation algorithm for most of these problems.

Table 1: Comparisons of 0.699-appro. algorithm and CMB

Pro.		SDP	0.699		CMB		
n	p	UB	F_Y	t_Y	F_C^*	t_C	σ
14	0.60	19.2	17	0.31	**19**	0.68	10^3
16	0.30	10.1	9	0.89	**10**	1.02	10^3
20	0.60	31.5	25	1.31	**31**	2.06	10^4
24	0.30	28.1	24	1.25	**27**	2.12	10^4
30	0.60	76.6	71	1.95	**75**	2.56	10^4
36	0.60	116.5	109	2.15	**114**	2.96	10^4
40	0.30	75.9	71	2.35	**75**	3.12	10^4

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一种求解最大二等分问题的连续化算法

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摘 要: 本文提出了一种求解最大二等分问题的连续化算法。我们首先将二等分问题转化为一个非线性规划；然后通过增广 Lagvange 罚函数方法来求解这个非线性规划问题。

关键词: 组合最优化；最大二等分；罚函数；NCP